Bounded from below Effective Potential for the $\mathcal{PT}$-Symmetric $\left(-\phi^4\right)$ Scalar Field Theory

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Abstract

Against the classical analysis concluding the instability of a bounded from above potential, we show that, up to two loops, the effective potential of the $\mathcal{PT}$-symmetric $\left(-\phi^4\right)$ scalar field theory is bounded from below as long as the vacuum condensate is pure imaginary. Unlike for the quantum mechanical case for the wrong sign $(-x^4)$ potential, this is the first time for the existence of an explanation of the vacuum stability of a bounded from above scalar field potential. Although the resulting effective Hamiltonian is non-Hermitian, it is $\mathcal{PT}$-symmetric as well as well-defined on the real line. With the proof of stability and in view of our recent work of ability to cure the Unitarity and Ghost states problems associated with the non-Hermitian representation of the theory (hep-th/0810.3687), a save employment of the theory to play the role of the Higgs Mechanism in the Standard model of Particle interactions is now possible.

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Among its wide range of applications, the subject of $\mathcal{PT}$-symmetric theories has stressed the bounded from above $(-x^4)$ quantum mechanical potential [1, 2, 3, 4, 5]. Such kind of theory has to be treated in a complex contour in the complex x-plane and demanding the vanishing of the wave functions as $|x| \rightarrow \infty$. Certainly, this boundary condition turns the problem to be non-Hermitian and $\mathcal{PT}$-symmetric as well. Besides, the positive definite metric operator for this theory and the equivalent Hermitian Hamiltonian have been obtained exactly [5]. Remarkably, the equivalent Hermitian Hamiltonian is bounded from below and thus gives no doubt that the spectrum is stable. On the other hand, for quantum field problems the treatment of the theory on a complex contour is hard to follow and may be impossible as it is possible to have radicals of the field. Moreover, we are prohibited to follow this route because replacing a real field with a complex one violates charge conservation. Indeed, an imaginary constant field shift (vacuum condensate) conserves the charge as well as play the role of a complex contour which lies inside the stock wedges. However, vacuum stability has not been discussed for the $(-\phi^4)$ scalar field theory and it seems that the only route to follow is the effective field approximate methods. In fact, exact treatments in quantum field theories exist only for models of little practical applications.

In a series of articles [6, 7, 8, 9], we have shown that the vacuum condensate of the $(-\phi^4)$ scalar field theory is pure imaginary and the effective theory obtained is well defined because of the presence of an imaginary ϕ^3 term which it has been shown that it is responsible for a real line square integrability of the wave functions in $0+1$ dimensions (quantum mechanics) [10, 11]. In higher dimensions, the amplitudes calculated with respect to the true vacuum can be converted to a calculation with respect to the free vacuum via the insertion of time evolution operator [12]. This picture is similar to the use of perturbation methods of the theory with a Harmonic oscillator basis in the quantum mechanical problems and thus the effective theory obtained is well defined and real line one. However, showing that the effective Hamiltonian of the $(-\phi^4)$ theory is well-defined and real line problem, is not sufficient to advocate the physical acceptability of the theory. A serious question that one may rise is to wonder if the theory is stable or not. In this work, we use the effective field approach to show that the $(-\phi^4)$ scalar field theory has a bounded from below effective potential and thus prove the stability of the theory in contradiction to the classical analysis which concludes the instability of any bounded from above potential.

Now consider the Hamiltonian density of the form

$$H=\frac{1}{2}\left((\nabla\phi)^{2}+\pi^{2}+m^{2}\phi^{2}\right)-\frac{g}{4}\phi^{4}, \tag{1}$$

where m is the mass of the one component scalar field ϕ and g is coupling constant. In applying the canonical transformation $\phi\rightarrow\psi+B$, where B is a constant called the vacuum condensate and there is no a priori constraint on B but we well show that for the theory to stable B has to be pure imaginary, we get the effective Hamiltonian

$$H=\frac{1}{2}\left((\nabla\psi)^{2}+\pi^{2}+m^{2}\left(\psi+B\right)^{2}\right)-\frac{g}{4}\left(\psi+B\right)^{4}. \tag{2}$$

Note that stability requires that the resulting ψ^{1} term vanishes [12].

In $1+1$ dimensions and up to two loops, one can obtain the effective potential of the theory as;

$$\frac{8\pi E}{m^{2}}=b^{2}-G\left(\frac{1}{4}b^{4}+\frac{3}{4}\ln^{2}t-\frac{3}{2}b^{2}\ln t\right)+\left(t-\ln t-1\right)+G^{2}\left(-3.515b^{2}\left(\frac{1}{t}-1\right)\right), \tag{3}$$

where $b^{2}=4\pi B^{2}$ and the dimensionless parameters $t=\frac{M^{2}}{m^{2}}$, $G=\frac{g}{2\pi m^{2}}$ have been used with M is the mass of the field ψ.

As a stability condition one always subject E to the condition $\frac{\partial E}{\partial b}=0$ [12] which yields the result;

$$-\frac{b}{t}\left(7.03G^{2}+tGb^{2}-3t\left(\ln t\right)G-2t\right)=0. \tag{4}$$

For $b\neq 0$, one can solve for t to have the result;

$$t=\exp\left(0.333\,33\frac{3.0\,\text{LambertW}\left(2.343\,3G\exp\left(-0.333\,33\frac{Gb^{2}-2.0}{G}\right)\right)G+Gb^{2}-2.0}{G}\right), \tag{5}$$

where LambertW is the Lambert's W function defined by $\text{LambertW}(x)e^{\text{LambertW}(x)}=x$. Substitute this result in E and for B imaginary we get the bounded from below effective potential plotted in Fig. 1. This result is pretty interesting as it is the first time to show that the vacuum of the $\mathcal{PT}$-symmetric $(-\phi^{4})$ scalar field theory is stable. Note that, with B imaginary the effective Hamiltonian obtained is non-Hermitian but $\mathcal{PT}$-symmetric and also the $B\psi^{3}$ term turns the theory well defined on the real line.

In $2+1$ dimensions and up to two loops, the effective potential can be obtained as

$$\frac{8\pi E}{m^{3}}=b^{2}-\frac{G}{12}b^{4}-\frac{1}{2}b^{2}G\left(1-t\right)+\frac{G^{2}}{6}\ln\left(t\right)\left(b^{2}+\left(1-t\right)\right)-\frac{G}{4}\left(1-t\right)^{2}+\frac{1}{3}\left(t^{3}-3t+2\right), \tag{6}$$

where $G = \frac{g}{4\pi m}$, $t = \frac{M}{m}$ and $b = B\sqrt{\frac{4\pi}{m}}$. Similarly, applying the condition condition $\frac{\partial E}{\partial b} = 0$ yields;

$$0 = \left\{1 - \frac{G}{6}\left(3\left(1-t\right)+b^2\right) + \frac{G^2}{6}\ln t\right\} b, \tag{7}$$

and for $b \neq 0$, we have

$$t = \exp\left(-\frac{1}{G}\left(G\,\mathrm{LambertW}\left(\frac{3}{G}\exp\left(\frac{1}{G}\left(-\frac{6}{G}+b^2+3\right)\right)\right)+\frac{6}{G}-b^2-3\right)\right). \tag{8}$$

Again, in substituting this result into the form of E, we obtain the bounded from below effective potential shown in Fig.2 as long as B is pure imaginary.

In conclusion, we show that while the classical potential is bounded from above and thus the classical analysis predicts instability of the spectrum, the quantum field effective potential is bounded from below and thus shows, for the first time, that the vacuum state for the $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory is stable. The stability of the theory is constrained by the existence of an imaginary condensate which turns the effective theory non-Hermitian but $\mathcal{PT}$-symmetric. In fact, the effective theory is well defined on the real line because of the existence of the pure imaginary $B\psi^3$ term in the Hamiltonian. With this in mind and in view of the new ansatz introduced by us in Ref.[13], one can cure the unitraity as well as ghost problems associated with the non-Hermitian representation of the theory.

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[1] Carl Bender and Stefan Boettcher, Phys.Rev.Lett.80:5243-5246 (1998).

[2] H. F. Jones and R. J. Rivers, Phys.Rev.D75:025023 (2007).

[3] Carl M. Bender, Dorje C. Brody and Hugh F. Jones, Phys.Rev.D73:025002(2006).

[4] H. F.Jones, J. Mateo and R. J. Rivers Phys.Rev.D74:125022 (2006).

[5] H. F. Jones, J. Mateo,Phys.Rev.D73:085002 (2006).

[6] Abouzeid M.Shalaby, Effective Field calculations of the Energy Spectrum of the $\mathcal{PT}$-Symmetric $(-x^4)$ Potential, hep-th/0810.3687 (to be published in Phys.Rev. D).

[7] Abouzeid M.Shalaby, Eur.Phys.J.C50:999-1006 (2007).

[8] Abouzeid Shalaby and M. EL-Houssieny, AIP Conf.Proc.1006:154-157 (2008).

[9] Abouzeid Shalaby, Czech.J.Phys.56:1033-1046 (2006).

[10] Jing-Ling Chen, L.C. Kwek and C.H.Oh, Phys. Rev. A 67, 012101 (2003).

[11] Carl M. Bender , Dorje C. Brody and Hugh F. Jones, Phys.Rev.D73:025002 (2006).

[12] Book by Michael E.Peskin and Daniel V.Schroeder, "AN INTRODUCTION TO THE QUANTUM FIELD THEORY" (1995).

[13] Abouzeid M.Shalaby, New Treatment of Ghost states and Unitarity problems in the PT-Symmetric $-\phi^4$ Scalar Field Theory, hep-th/0901.3919 (2009).

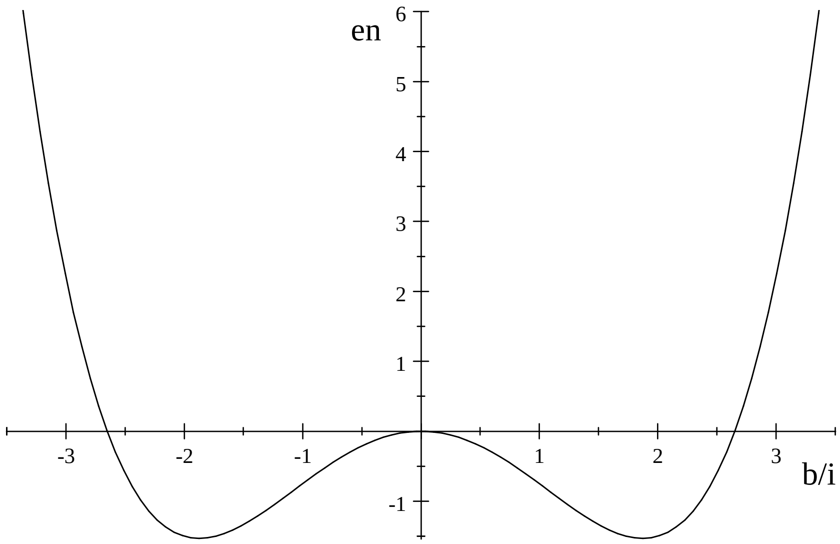

FIG. 1: The effective potential $en = \frac{8\pi E}{m^2}$ versus the vacuum condensate b for $G = \frac{1}{2}$ for the $\mathcal{PT}$-symmetric$\left(-\phi^4\right)$ scalar field theory in $1+1$ space-time dimensions.

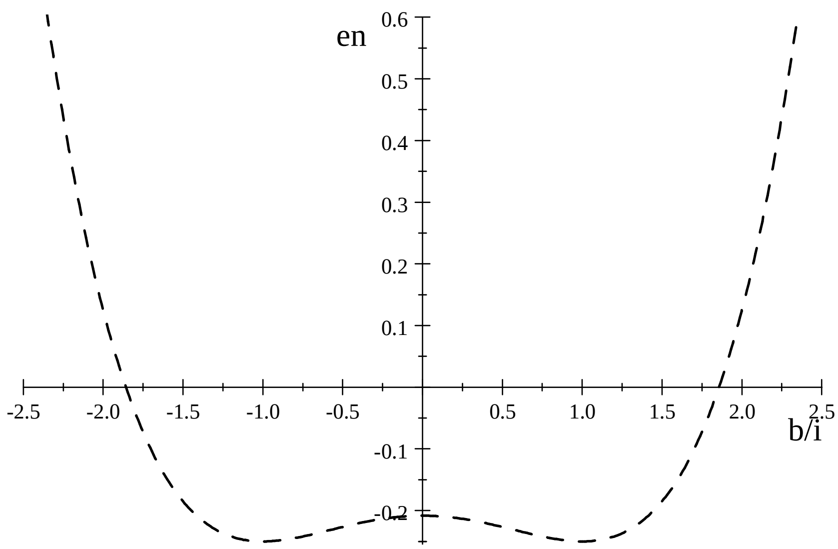

FIG. 2: The effective potential $en = \frac{8\pi E}{m^3}$ versus the vacuum condensate b for $G = \frac{1}{2}$ for the $\mathcal{PT}$-symmetric$\left(-\phi^4\right)$ scalar field theory in $2+1$ space-time dimensions.